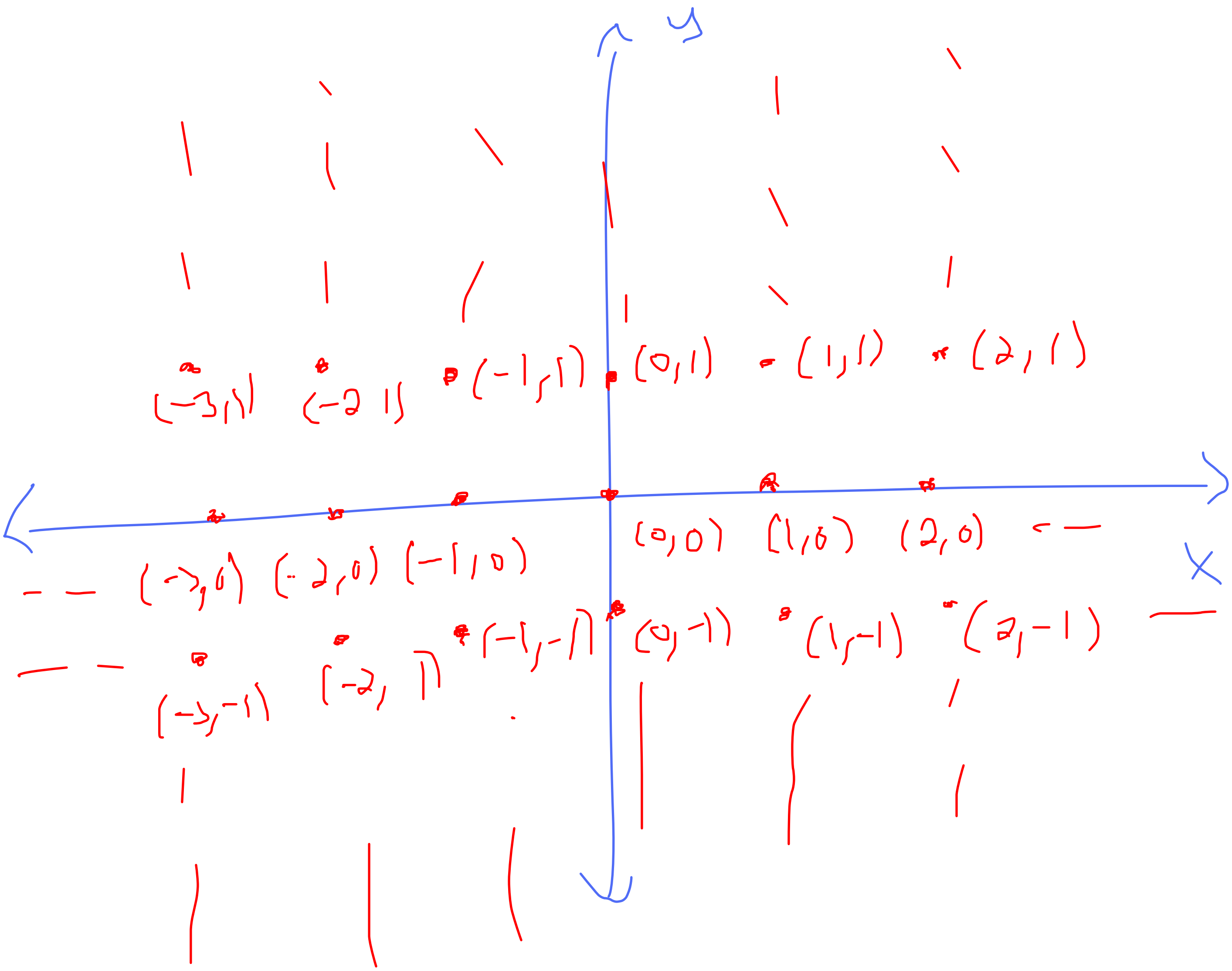
Recall: We showed that the countable union of countable sets is countable. Actually, we showed the disjoint union of countable sets is countable. We also showed every subset of a countable set is countable.

Theorem: $\mathbb{Q}$ is countable

Before the proof:

The picture: Identify $\mathbb{Q}$ with a subset of the integer lattice in $\mathbb{R}^2$

Integer lattice = all points $(m,n) \in \mathbb{R}^2$ such that $m,n \in \mathbb{Z}$



For each fixed $n \in \mathbb{Z}$, $\{(n,m) : m \in \mathbb{Z}\}$ has countable cardinality.

Furthermore, for a fixed $k \in \mathbb{Z}$, if $k \neq n$,

then

$$\{(n,m) : m \in \mathbb{Z}\} \cap \{(k,m) : m \in \mathbb{Z}\} = \emptyset.$$

The integer lattice $\mathbb{Z}^2$ is then expressible as the countable union of disjoint sets,

$$\mathbb{Z}^2 = \bigcup_{n \in \mathbb{Z}} \{(n,m) : m \in \mathbb{Z}\}.$$

We then have $\mathbb{Z}^2$ expressed as the countable union of countable sets, and so $\mathbb{Z}^2$ is countable.

Proof:

We define

$$\varphi : \mathbb{Q} \to \mathbb{Z}^2,$$

$\varphi(0) = (0,0)$ and if $n, m \in \mathbb{Z}$, $\frac{n}{m}$ is in lowest terms, define

$$\varphi\left(\frac{n}{m}\right) = (n, m)$$

Let's show φ is injective.

Suppose $\varphi\left(\frac{n}{m}\right) = \varphi\left(\frac{i}{k}\right)$

for some $n, m, i, k \in \mathbb{Z}$,

$m, k \neq 0$.

If either $\varphi\left(\frac{n}{m}\right)$ or $\varphi\left(\frac{i}{k}\right)$

is $(0,0)$, then either n

is zero or i is zero

(respectively), which then

forces the other to be zero,

so $\frac{n}{m} = \frac{i}{k} = 0$.

Now suppose

$$\varphi\left(\frac{n}{m}\right) = \varphi\left(\frac{i}{k}\right) \neq (0,0)$$

for $n, m, i, k \in \mathbb{Z}$, $m, k \neq 0$.

Then $(n,m) = \varphi\left(\frac{n}{m}\right) = \varphi\left(\frac{i}{k}\right) = (i,k)$

$\Rightarrow$ $n = i$, $m = k$, so

$$\frac{n}{m} = \frac{i}{k}.$$

(we need the lowest terms assumption here)

So φ is injective, hence $\mathbb{Q}$ is in bijective correspondence with a subset of a countable set, hence, $\mathbb{Q}$ is countable. $\square$

Lemma: $(0,1)$ is not countable.

proof: Cantor diagonalization;

by contradiction.

We suppose that $\exists$ a bijection

$\varphi: \mathbb{N} \to (0,1)$.

Each number in $(0,1)$ has

a base-10 decimal expansion

We can assume that all

expansions are infinite, using

the following:

Fact: $1 = .\overline{9}$

Why? Well, $\frac{1}{3} = .\overline{3}$

$$1 = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = .\overline{3} + .\overline{3} + \overline{3}$$
$$= .\overline{9}$$

If $x \in (0,1)$ has a finite base 10 expansion,

$$x = .x_1 x_2 \cdots x_n \quad (x_i \in \{0, 1, \ldots, 9\})$$

write

$$x = .x_1 x_2 \cdots (x_n - 1)\overline{9}$$

write

$$\varphi(1) = .x_{1,1}\, x_{1,2}\, x_{1,3} \cdots$$

$$\varphi(2) = .x_{2,1}\, x_{2,2}\, x_{2,3} \cdots$$

$$\varphi(3) = .x_{3,1}\, x_{3,2}\, x_{3,3} \cdots$$

$$\vdots$$

$$x_{i,j} \in \{0, 1, 2, \ldots, 9\}$$

$$\forall\ 1 \leq i, j < \infty$$

Define $y \in (0,1)$ by

$$y = .y_1\, y_2\, y_3\, y_4 \cdots$$

where

$$y_i = \begin{cases} x_{i,i}+1 & \text{if } x_{i,i} \neq 9 \\ 0 & \text{if } x_{i,i} = 9 \end{cases}$$

Then $y_1 \neq x_{1,1}$, so $y \neq \varphi(1)$,

$y_2 \neq x_{2,2}$, so $y \neq \varphi(2)$,

$y_3 \neq x_{3,3}$, so $y \neq \varphi(3)$

and generally, $y_n \neq x_{n,n}$, so $y \neq \varphi(n)$ for any $n \in \mathbb{N}$.

This is a contradiction since $y \in (0,1)$ and we assumed φ was bijective. Therefore, $(0,1)$ is not countable. $\square$

Natural Question: We know the cardinality of $[0,1)$ is "greater than" the cardinality of $\mathbb{N}$. Is there a set whose cardinality is strictly "in between" these two cardinalities? The existence of such a set is called the **Continuum Hypothesis**.

In fact, Gödel proved that the existence of such a set is consistent with the usual axioms of set theory (+ axiom of choice).

Cohen proved that the non-existence of such a set is consistent with the usual axioms!

So The continuum hypothesis can't be proved or disproved using the usual axioms!